On Oscillatory Integrals in Higher Dimension

J. Bourgain

Some problems in Harmonic analysis **(E. Stein)**

(Proc. Symp. Pure Math, 1979)

Restriction Problem

For which values of p and q does the apriori inequality

$$\left[\int_{|\xi|=1} |\widehat{f}(\xi)|^q \sigma(d\xi)\right]^{1/q} \leq C\|f\|_p, \qquad f \in \mathcal{S}(\mathbb{R}^n)$$

hold?

Conjecture: $1 < p < \frac{2n}{n+1}, q = \frac{n-1}{n+1}p'$

$\underline{n = 2}$ Proven by Fefferman, Zygmund

$\underline{n \geq 2}$ $q = 2, p = \frac{2(n+1)}{n+3}$ (Stein, Tomas)

Only partial results in the range $\frac{2(n+1)}{n+3} < p < \frac{2n}{n+1}$

Bochner–Riesz Problem

Define spherical Fourier multipliers

$$(S_\delta f)^\wedge(\xi) = (1 - |\xi|^2)^\delta_+ \ \hat{f}(\xi) \qquad f \in \mathcal{S}(\mathbb{R}^n),\ n \geq 2$$

What are the mapping properties of S_δ?

Fefferman: For $p \neq 2$, S_0 is unbounded on $L^p(\mathbb{R}^n)$.

First appearance of the connection with Besicovitch-Kakeya sets

Conjecture: S_δ *bounded on* $L^p(\mathbb{R}^n)$ *provided* $\delta > 0$ *and*

$$\frac{2n}{n+1+2\delta} < p < \frac{2n}{n-1-2\delta}$$

$$\widehat{S}_\delta(x) \sim \frac{e^{2\pi i|x|}}{|x|^{\frac{n+1}{2}+\delta}}$$

$\underline{n = 2}$ Proven by Carleson-Sjölin, Hormander

$\underline{n \geq 2}$ $\max(p, p') \geq \frac{2(n+1)}{n-1}$

Beyond the L^2-theory $\quad (n \geq 2)$

$\underline{n=3}$	B	(91)	$p' > 4 - \frac{2}{15}$
	Wolff	(95)	$p' > 4 - \frac{2}{11}$
	Tao-Vargas-Vega	(98)	$p' > 4 - \frac{2}{9}$
	Tao-Vargas	(00)	$p' > 4 - \frac{2}{7}$
	Tao	(03)	$p' > 3\frac{1}{3}$
	B-Guth	(10)	$p' > 3\frac{3}{10}$

Conjectured $p' > 3 \Rightarrow$ Kakeya Conjecture

n arbitrary

Stein-Thomas (75) $p' \geq \frac{2(n+1)}{n-1} = 2 + \frac{4}{n-1}$

Tao (03) $p' > \frac{2(n+2)}{n} = 2 + \frac{4}{n}$

B-Guth $(n \geq 4)$

$$\begin{cases} n \equiv 0 (\bmod 3) & p' > 2\frac{4n+3}{4n-3} = 2 + \frac{12}{4n-3} \\ n \equiv 1 (\bmod 3) & p' > \frac{2n+1}{n-1} \\ n \equiv 2 (\bmod 3) & p' > \frac{4(n+1)}{2n-1} \end{cases}$$

Conjectured $p' > \frac{2n}{n-1} = 2 + \frac{2}{n-1}$

General Setting

$$(T_\lambda f)(x) = \int e^{i\lambda\psi(x,y)} f(y)dy \qquad (\lambda \to \infty)$$

$x \in \mathbb{R}^n, y \in \mathbb{R}^{n-1}$ restricted to neighborhoods of 0

real analytic phase functions

$$\psi(x,y) = x_1y_1 + \cdots + x_{n-1}y_{n-1} + x_n\langle Ay, y\rangle + 0(|x|\,|y|^3) + 0(|x|^2|y|^2)$$

A is a non-degenerate $(n-1) \times (n-1)$ matrix

Problem: What are the mapping properties of T_λ?

Exponents (r, q) such that

$$\|T_\lambda f\|_q \leq C\lambda^{-n/q}\|f\|_r$$

$\underline{n = 2}$ Hormander (73) $q > \frac{2n}{n-1}$ and $\frac{n+1}{(n-1)q} + \frac{1}{r} \leq 1$

$\underline{n \geq 2}$ Stein (87) $r = 2, q = \frac{2(n+1)}{n-1}$

Distinction between even and odd dimension

n odd

Inequality

$$\|T_\lambda f\|_q \le C\lambda^{-n/q}\,\|f\|_\infty$$

may only hold for $q \ge \frac{2(n+1)}{n-1}$

Example $(n=3)$

$$\psi(x,y) = x_1y_1 + x_1y_2 + 2x_3y_1y_2 + x_3^2y_1^2 \qquad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$f(y) = e^{i\lambda y_2^2}$$

$$\int e^{i\lambda\psi(x,y)} f(y)dy = \int e^{i\lambda\left(x_2(x_3y_1+y_2)+(x_3y_1+y_2)^2\right)+(x_1-x_2x_3)y_1\big)}$$

$$|x_1 - x_2x_3| < \frac{1}{\lambda} \Rightarrow |(T_\lambda f)(x)| \sim \frac{1}{\sqrt{\lambda}}$$

$\Rightarrow$ condition $\quad (\frac{1}{\lambda})^{\frac{1}{2}+\frac{1}{q}} \leq (\frac{1}{\lambda})^{\frac{3}{q}} \Rightarrow q \geq 4$

n even

Theorem (B-Guth)

$$\|T_\lambda f\|_q \leq C\lambda^{-n/q}\|f\|_\infty$$

for

$$q > \frac{2(n+2)}{n}$$

Result is best possible if no further assumptions

Example $(n = 4)$

$$\psi(x,y) = x_1y_1 + x_2y_2 + x_3y_3 + 2x_4(y_1y_2 + y_3^2) + x_4^2\,y_1^2$$

$$f(y) = e^{i\lambda y_2^2}$$

Distinction between definite and indefinite quadratic forms

$$\psi(x,y) = x_1y_1 + \cdots + x_{n-1}y_{n-1} + x_n\langle Ay, y\rangle + 0(|x|\,|y|^3) + 0(|x|^2|y|^2)$$

Theorem (B-Guth)

Assume A is positive (or negative) definite. Then

$$\|T_\lambda f\|_q \leq C\lambda^{-n/q}\|f\|_\infty$$

holds, if

$$\begin{cases} q > 2\frac{4n+3}{4n-3} & \textit{if} \quad n \equiv 0 \pmod 3 \\ q > \frac{2n+1}{n-1} & \textit{if} \quad n \equiv 1 \pmod 3 \\ q > \frac{4(n+1)}{2n-1} & \textit{if} \quad n \equiv 2 \pmod 3 \end{cases}$$

In particular, for $n = 3, q > \frac{10}{3}$ is optimal

Role of (curved) Kakeya-type sets

$$\psi(x,y) = x_1 y_1 + \cdots + x_{n-1} y_{n-1} + x_n \langle Ay, y\rangle + \psi_1(x,y)$$

$A = A^*$ non-degenerate

For $y, \omega \in B^{n-1}$, let

$$\Gamma_y(\omega) = \{x \in B^n; \nabla_y \psi(x,y) = \omega\}$$

which is a curve

$$x' = \omega - x_n A y + \nabla_y \psi_1 \qquad x' = (x_1, \ldots, x_{n-1})$$

Definition

A compact subset $E \subset \mathbb{R}^n$ *is a* ψ*-Kakeya set if for all* $y \in B^{n-1}$*, there is some* $\omega \in B^{n-1}$ *with* $\Gamma_y(\omega) \subset E$

For $\delta > 0$, let

$$T_y^\delta(\omega) = \{x \in B^n ; |\nabla_y \psi(x,y) - \omega| < \delta\}$$

and define associated maximal operator

$$\mathcal{M}_\delta F(y) = \sup_{\omega \in B^{n-1}} \frac{1}{|T_y(\omega)|} \int_{T_y(\omega)} |F(x)| dx$$

Problem 1

What is the minimal dimension of a ψ-Kakeya set?

$$\Big(\text{always at least } \frac{n+1}{2}\Big)$$

Problem 2

For which p does the inequality

$$\|\mathcal{M}_\delta\|_{p\to p} \lesssim \left(\frac{1}{\delta}\right)^{\frac{n}{p}-1}$$

hold?

$$\|T_\lambda f\|_q \lesssim \lambda^{-n/q}\|f\|_\infty \qquad (*)$$

$$\Downarrow$$

$$\dim \psi\text{-Kakeya set } \geq 2\left(\frac{q}{2}\right)' - n$$

In particular, if $(*)$ holds for all $q > \frac{2n}{n-1}$, then $\dim = n$

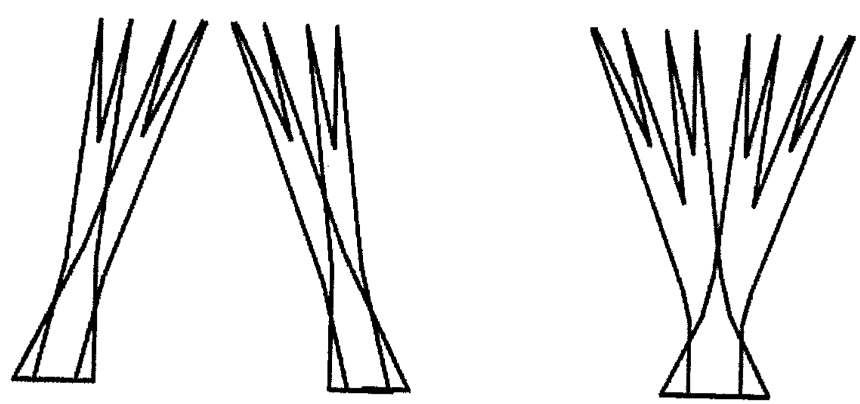
The restriction case (ψ linear in x)

Usual notion of '**Kakeya Set**':

$E \subset \mathbb{R}^n$ containing a line segment in every direction

Conjecture Such sets are always of maximal dimension $(\dim E = n)$

Proven for $n = 2$ (Davies) but may be of zero-Lebesque measure (Besicovitch)



$n \geq 3$ only partial results

$$n = 3 \quad \dim \geq \frac{5}{2} + \varepsilon$$ (Katz-Laba-Tao)

$$n \geq 4 \quad \dim \geq (1 - \sqrt{2})(n - 4) + 3$$ (Katz-Tao)

Extensive research that generated other developments

Arithmetic combinatorics (sum-product phenomena)

Number theory and group theory (bounds on exponential sums, expansion and spectral gaps in linear groups)

Theoretical computer science (Z. Dvir's proof of finite fields Kakeya conjecture)

Dimension of curved Kakeya sets

$\underline{n \text{ odd}}$ Worst case scenario may occur

$$\dim E = \frac{n+1}{2}$$

$\underline{n = 3}$

Indefinite case

$$\psi(x,y) = x_1 y_1 + x_2 y_2 + 2x_3 y_1 y_2 + x_3^2 y_1^2$$

$$\Gamma_y : \nabla_y \psi + \begin{pmatrix} 0 \\ 2y_2 \end{pmatrix} = 0 \Rightarrow \begin{cases} x_1 = -2y_2 x_3 - 2y_1 x_3^2 \\ \\ x_2 = -2y_1 x_3 - 2y_2 \end{cases}$$

$$\Rightarrow \quad x_1 = x_2 x_3$$

$$\|T_\lambda f\|_q \le C\lambda^{3/q} \|f\|_\infty \quad \text{for } \; q \ge 4 \text{ only}$$

Definite case

$$\psi(x,y) = -x_1y_1 - x_2y_2 + \frac{1}{2}x_3(y_1^2 + y_2^2) + x_3^2y_1y_2 + \frac{1}{2}x_3^3y_2^2$$

$$\Gamma_y : \nabla_y\psi - \begin{pmatrix} y_2 \\ 0 \end{pmatrix} = 0 \Rightarrow \begin{cases} x_1 = y_1x_3 + y_2x_3^2 + y_2 \\ \\ x_2 = y_2x_3 + y_1x_3^2 + y_2x_3^3 \end{cases}$$

$$\Rightarrow x_2 = x_1x_3$$

$$\|T_\lambda f\|_q \leq C\lambda^{3/q}\|f\|_\infty \text{ for } q > 10/3 \text{ (optimal)}$$

$$2 = \dim E = 2\left(\frac{q}{2}\right)' - 3$$

n even

Theorem (B-Guth)

Curved Kakeya sets are always at least of

$\dim \geq \frac{n}{2} + 1$

(optimal)

Main tool

Multilinear theory (Bennett, Carbery, Tao; 2006)

$$\psi(x,y) = x_1 y_1 + \cdots + x_{n-1} y_{n-1} + x_n \langle Ay, y\rangle + \psi_1(x,y)$$

For $1 \le j \le k \le n$, let $U_j \subset \mathbb{R}^{n-1}$ be disjoint balls and

$$\phi_j = \psi\Big|_{y \in U_j}$$

Define

$$Z_j = \partial_{y_1}(\nabla_x \phi_j) \wedge \cdots \wedge \partial_{y_{n-1}}(\nabla_x \phi_j)$$

Assume transversality condition

$$|Z_1(x, y^{(1)}) \wedge \cdots \wedge Z_k(x, y^{(k)})| > C \quad \text{for} \quad y^{(j)} \in U_j$$

Theorem 1 (B-C-T)

$$T_\lambda^{(j)} f(x) = \int_{U_j} e^{i\lambda\psi_j(x,y)} f(y)dy$$

$$\Big\| \Big(\prod_{j=1}^{k} |T_\lambda^{(j)} f_j| \Big)^{\frac{1}{k}} \Big\|_q \ll \lambda^{-\frac{n}{q}+\varepsilon} \Big(\prod_{j=1}^{k} \|f_j\|_2 \Big)^{\frac{1}{k}}$$

if $q \geq \frac{2k}{k-1}$

Theorem 2 (B-C-T)

Let $\delta > 0$ and for $j = 1, \dots, k$, let $\mathcal{T}^{(j)}$ be a collection of curved tubes

$$\Gamma_{y,\omega} = \{x \in B^n; |\nabla_y \psi^{(j)}(x,y) - \omega| < \delta\} \text{ with } y \in U_j$$

Then, for $q = \frac{k}{k-1}$

$$\Big\| \prod_{j=1}^{k} \Big(\sum_{\Gamma \in \mathcal{T}^{(j)}} 1_\Gamma \Big)^{1/k} \Big\|_q \ll \delta^{\frac{n}{q}-\varepsilon} \prod_{j=1}^{k} (\#\mathcal{T}^{(j)})^{1/k}$$

Sketch of Method

Take $d = 3$ and consider model case

$$\phi(x, y) = x_1 y_1 + x_2 y_2 + x_3(y_1^2 + y_2^2)$$

$$Tf(x) = \int_{\text{loc}} e^{i\phi(x,y)} f(y)dy$$

We prove the inequality

$$\|Tf\|_{L^q(B_R)} \leq R^{\varepsilon}\|f\|_{\infty} \text{ for } q > \frac{10}{3}$$

using multi-linear theory.

From ε-removal lemmas, it implies

$$\|Tf\|_{L^q(\mathbb{R}^3)} \leq C_q\|f\|_q \text{ for } q > \frac{10}{3}$$

Refinements of the method permit to reach $q > 3\frac{3}{10}$

$\Omega \subset \mathbb{R}^2$ fixed neighborhood of 0

Fix large parameters $1 \ll K_1 \ll K$

Partition $\Omega = \bigcup \Omega_\alpha$ in K^2 boxes of size $\frac{1}{K}$

$$Tf(x) = \sum_\alpha e^{i\phi(x,y_\alpha)} \Big[\int_{\Omega_\alpha} e^{i[\phi(x,y)-\phi(x,y_\alpha)]} f(y)dy \Big]$$

$$= \sum_\alpha e^{i\phi(x,y_\alpha)} (T_\alpha f)(x)$$

$$|\nabla_x [\phi(x,y) - \phi(x,y_\alpha)]| < \frac{1}{K}$$

$\Rightarrow |T_\alpha f| \sim$ constant on $\mathbb{R}^3$-balls of size K

Fix K-ball B_K and write

$$|T_\alpha f(x)| \sim c_\alpha \text{ for } x \in B_K$$

Let

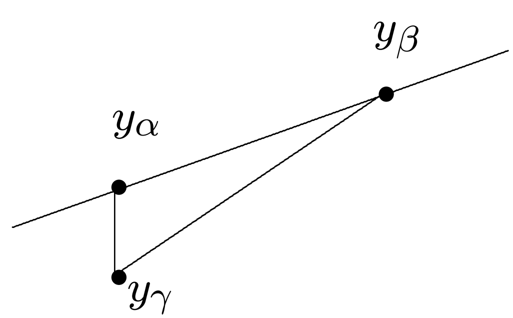
$$c_* = \max_\alpha c_\alpha$$

There are 3 alternatives

(I) Non-coplanar interaction

There are α, β, γ such that $c_\alpha, c_\beta, c_\gamma > K^{-4} c_*$ and

$$|y_\alpha - y_\beta| \geq |y_\alpha - y_\gamma| \geq \text{dist}\left(y_\gamma, y_\alpha + \mathbb{R}(y_\beta - y_\alpha)\right) > \frac{10^3}{K}$$



(II) Non-transverse interaction

$$|y_\alpha - y_{\alpha *}| > \frac{1}{K_1} \Rightarrow c_\alpha < K^{-4} c_*$$

(III) Transverse coplanar interaction

Failure of (I) and (II)

Case (I)

Use **trilinear theory.** For $x \in B_K$

$$|Tf(x)| \le \sum_\alpha c_\alpha < K^2 c_* < K^6 (c_\alpha c_\beta c_\gamma)^{\frac{1}{3}}$$

and for $q > 3$

$$|Tf(x)|^q \le |Tf(x)|^3 <$$

$$K^{18} \sum_{\alpha,\beta,\gamma(I)} |T_\alpha f|(x) |T_\beta f|(x) |T_\gamma f(x)$$

From [BCT]

$$\int_{B_R} |T_\alpha f|.|T_\beta f|.|T_\gamma f| \ll R^\varepsilon \, C(K)$$

$\Rightarrow$ collected contribution

$$\int_{B_R} |Tf|^q \ll R^\varepsilon$$

Case (II)

Use **parabolic rescaling.** For $x \in B_K$

$$|Tf(x)| \leq \left| \int_{B(y_{\alpha*}, \frac{1}{K_1})} e^{i\phi(x,y)} f(y) dy \right| + K^{-2} c_*$$

and

$$|Tf(x)|^q \leq \sum_{\tau < K_1^2} \left| \int_{\tilde{\Omega}_\tau} e^{i\phi(x,y)} f(y) dy \right|^q + K^{-2q} \sum_{\alpha < K^2} \left| \int_{\Omega_\alpha} e^{i\phi(x,y)} f(y) dy \right|^q$$

Set

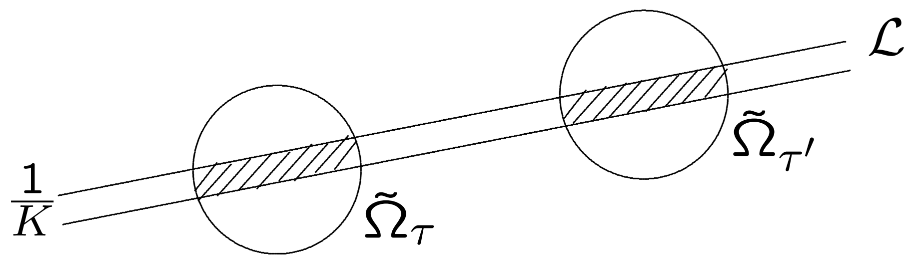
$$Q = \max_{|f| \leq 1} \|Tf\|_{L^q(B_R)}$$

Using rescaling, Case II contribution is at most

$$K_1^2 K_1^{-2q} K_1^4 Q^q + K^{-2q} K^2 K^{-2q} K^4 Q^q < K_1^{2(3-q)} Q^q$$

Case III

Use **square function inequalities**



For $x \in B_K$, $|Tf(x)| < \left| \sum_{\Omega_\alpha \subset \mathcal{L}} \int_{\Omega_\alpha} e^{i\phi(x,y)} f(y) dy \right| +$ negligible

$$< K_1^3 \left| \int_{\tilde{\Omega}_\tau \cap \mathcal{L}} e^{i\phi(x,y)} f(y) dy \right|^{\frac{1}{2}} . \left| \int_{\tilde{\Omega}_{\tau'} \cap \mathcal{L}} e^{i\phi(x,y)} f(y) dy \right|^{\frac{1}{2}}$$

(unless Case II occurs)

$$\|Tf\|_{L^q(B_K)} < K^{3(\frac{1}{q}-\frac{1}{4})} \|Tf\|_{L^4_{(B_R)}} < C(K_1) K^{3(\frac{1}{q}-\frac{1}{4})} K^{3/4} \Big(\sum_{\Omega_\alpha \subset \mathcal{L}} |T_\alpha f|^2 \Big)^{\frac{1}{2}}$$

$$< C(K_1) K^{3(\frac{1}{q}-\frac{1}{4})} K^{\frac{3}{4}}\, K^{\frac{1}{2}-\frac{1}{q}} \Big(\sum_\alpha |T_\alpha f|^q \Big)^{\frac{1}{q}}$$

Collected contribution from Case III, by rescaling

$$C(K_1)\, K^{\frac{1}{2}-\frac{1}{q}} K^{\frac{2}{q}}\, K^{-2} K^{\frac{4}{q}} Q < C(K_1)\, K^{\frac{5}{q}-\frac{3}{2}} Q$$

This proves that

$$Q < R^\varepsilon + K_1^{2(\frac{3}{q}-1)}\, Q + C(K_1)\, K^{\frac{5}{q}-\frac{3}{2}}\, Q$$

$\Rightarrow\ Q < R^\varepsilon$ for $q > \frac{10}{3}$

Discretized ring theorem in $\mathbb{R}$

Notation: For $A \subset \mathbb{R}$, $\delta > 0$, denote $N(A, \delta)$ the minimum number of δ-balls needed to cover A

Theorem *Given* $0 < \alpha < 1$ *and* $\varepsilon > 0$*, there is* $\beta > \alpha$ *such that for* $\delta > 0$ *small enough and* $A \subset [0, 1]$ *satisfying*

$$N(A, \delta) > \delta^{-\alpha}$$

$$N(A \cap I_\rho, \delta) < \rho^\varepsilon N(A, \delta) \quad \textit{for } \delta < \rho < 1, I_\rho = \rho\textit{-interval}$$

either

$$N(A + A, \delta) > \delta^{-\beta}$$

or

$$N(A.A, \delta) > \delta^{-\beta}$$

Application to spectral gaps and expansion in group theory

Theorem *Let* $G = SU(d), d \geq 2$ *and* $g_1, \ldots, g_k \in G$ *be algebraic elements generating a subgroup* $\langle g_1, \ldots, g_k \rangle$ *of* G *which is topologically dense.*

Define the Hecke operator T *on* $L^2(G)$

$$Tf(x) = \sum_{1 \leq j \leq k} \Big(f(g_j x) + f(g_j^{-1} x) \Big)$$

There is $\gamma > 0$ *such that for* $f \in L^2(G)$, $\int f = 0$

$$\|Tf\|_2 \leq (1 - \gamma)\|f\|2$$

Applications to geometry (aperiodic tilings) and to quantum computation (the Solovay-Kitaev algorithm)